2806. [2003 : 45] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that x, y, $z > 0$, $\alpha \in \mathbb{R}$ and $x^\alpha + y^\alpha + z^\alpha = 1$. Prove that

(a) $x^2+y^2+z^2 \ \geq\ x^{\alpha+2}+y^{\alpha+2}+z^{\alpha+2}+2x^2y^2z^2\left(x^{\alpha-2}+y^{\alpha-2}+z^{\alpha-2}\right)$,

(b) $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2} \ \geq\ x^{\alpha-2}+y^{\alpha-2}+z^{\alpha-2}+\dfrac{2\left(x^{\alpha+1}+y^{\alpha+1}+z^{\alpha+1}\right)}{xyz}$.

Solution by Arkady Alt, San Jose, CA, USA; David Loeffler, student, Trinity College, Cambridge, UK; and Panos E. Tsaoussoglou, Athens, Greece.

(a) We will show that the proposed inequality is true for $\alpha > 0$ and the reverse inequality holds for $\alpha < 0$. For $\alpha = 0$, the given condition $x^\alpha + y^\alpha + z^\alpha = 1$ is never satisfied.

Let D be the difference between the left side and the right side of the given inequality. Using $x^\alpha + y^\alpha + z^\alpha = 1$, we obtain

$$\begin{aligned}
D &= (x^2+y^2+z^2) - x^{\alpha+2} - y^{\alpha+2} - z^{\alpha+2} \\
&\qquad - 2x^\alpha y^2z^2 - 2y^\alpha z^2x^2 - 2z^\alpha x^2y^2 \\
&= (x^2+y^2+z^2)(x^\alpha+y^\alpha+z^\alpha) - x^{\alpha+2} - y^{\alpha+2} - z^{\alpha+2} \\
&\qquad - 2x^\alpha y^2z^2 - 2y^\alpha z^2x^2 - 2z^\alpha x^2y^2 \\
&= x^\alpha(y^2+z^2) - 2x^\alpha y^2z^2 + y^\alpha(z^2+x^2) \\
&\qquad - 2y^\alpha z^2x^2 + z^\alpha(x^2+y^2) - 2z^\alpha x^2y^2 \\
&= x^\alpha[y^2(1-z^2)+z^2(1-y^2)] + y^\alpha[z^2(1-x^2)+x^2(1-z^2)] \\
&\qquad + z^\alpha[x^2(1-y^2)+y^2(1-x^2)]\,.
\end{aligned}$$

Now, if $\alpha > 0$, then the condition $x^\alpha + y^\alpha + z^\alpha = 1$ implies that $x < 1$, $y < 1$, and $z < 1$, so that $D \geq 0$ and the inequality holds. If $\alpha < 0$, then $x > 1$, $y > 1$, and $z > 1$ and the reverse inequality holds.

(b) The desired inequality follows easily by applying the condition $x^\alpha + y^\alpha + z^\alpha = 1$ and the AM–GM Inequality:

$$\begin{aligned}
\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} &= \left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)\cdot(x^\alpha+y^\alpha+z^\alpha) \\
&= x^{\alpha-2}+y^{\alpha-2}+z^{\alpha-2}+x^\alpha\left(\frac{1}{y^2}+\frac{1}{z^2}\right) \\
&\qquad + y^\alpha\left(\frac{1}{z^2}+\frac{1}{x^2}\right) + z^\alpha\left(\frac{1}{x^2}+\frac{1}{y^2}\right) \\
&\geq x^{\alpha-2}+y^{\alpha-2}+z^{\alpha-2} \\
&\qquad + x^\alpha\left(\frac{2}{yz}\right) + y^\alpha\left(\frac{2}{zx}\right) + z^\alpha\left(\frac{2}{xy}\right) \\
&= x^{\alpha-2}+y^{\alpha-2}+z^{\alpha-2}+\frac{2(x^{\alpha+1}+y^{\alpha+1}+z^{\alpha+1})}{xyz}\,.
\end{aligned}$$